Averaged models for compressible multiphase flows

Séminaire des doctorant·es de l'ICJ et de l'UMPA

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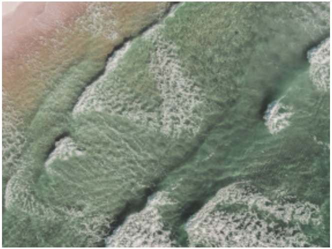
Outline

Outline

1. About averaged models
   - Motivations
   - Different types of averaged models
   - Averaged models for compressible multiphase flows

Why averaged models ?

Modelizing fluid mechanics



Why averaged models ?

Macroscopic models

- The Navier-Stokes equations describe the flow on a microscopic scale.
- Problem(s) : mathematically unsolvable, computations of approximate solutions cost too much, inconsistent with industrial needs of a macroscopic description of the flow.

[1] M. Ishii and T. Hibiki. *Thermo-Fluid Dynamics of Two-Phase Flow*. 2nd ed. New York: Springer, 2011.

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Solution : averaged models.[1]

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Different types of averaged models

Statistical average

- **Type of situation:** bubbly flows
- **Idea:** ensemble = set of possible motions (or realizations of a motion). Define a probability measure on the ensemble and integrate with respect to that measure.[2]

Figure: Bubbly flow

[2] D. A. Drew and S. L. Passman. *Theory of Multicomponent Fluids*. Ed. by J. E. Marsden and L. Sirovich. Vol. 135. Applied Mathematical Sciences. Springer, 1999.

Different types of averaged models

Homogeneization

- **Type of situation:** high-frequency alternance of thin straps
- **Idea:** take the limit when $\varepsilon \to 0$ of the equations for the two fluids (with a highly oscillating characteristic function) to get one set of equations to describe the mixture. This technique requires Dirichlet boundary conditions and the introduction of Young measures.[3]

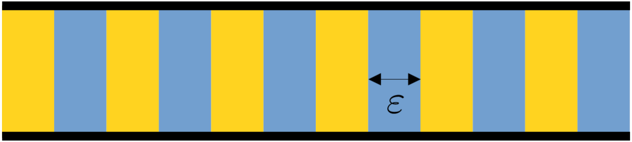

Figure: Mesoscopic mixture before homogeneisation process

[3] D. Bresch, C. Burtea, and F. Lagoutière. "Mathematical justification of a compressible bifluid system with different pressure laws: a continuous approach". In: *Applicable Analysis* 101.12 (Aug. 2022), pp. 4235–4266.

Different types of averaged models

Space averaging (dimension reduction)

- **Type of situation:** thin domains
- **Idea:** one dimension of the problem is much smaller than the others. Approximation obtained by averaging the Navier-Stokes equations over this dimension.[4][5]

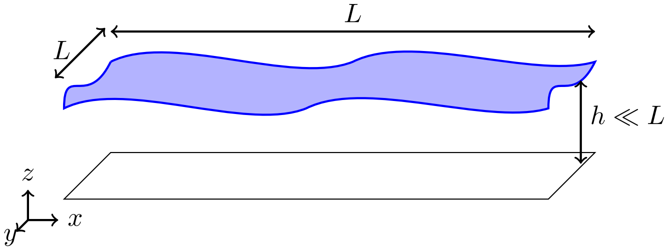

Figure: Shallow water configuration

[4] H. Bruce Stewart and B. Wendroff. "Two-phase flow: Models and methods". In: *Journal of Computational Physics* 56.3 (Dec. 1984), pp. 363–409.

[5] J.-F. Gerbeau and B. Perthame. "Derivation of Viscous Saint-Venant System for Laminar Shallow Water; Numerical Validation". In: *Discrete & Continuous Dynamical Systems - B* 1.1 (2001), pp. 89–102.

The Baer-Nunziato system for multiphase flows

The Baer-Nunziato[6] model for a bifluid flow is :

$$\begin{cases} \partial_t \alpha_1 + v_i \partial_x \alpha_1 & = & \theta_p (p_1 - p_2), \\ \partial_t(\alpha_1\rho_1) + \partial_x(\alpha_1\rho_1 u_1) & = & 0, \\ \partial_t(\alpha_1\rho_1 u_1) + \partial_x(\alpha_1\rho_1 u_1^2 + \alpha_1 p_1) - p_i \partial_x \alpha_1 & = & \theta_u (u_2 - u_1), \\ \partial_t(\alpha_2\rho_2) + \partial_x(\alpha_2\rho_2 u_2) & = & 0, \\ \partial_t(\alpha_2\rho_2 u_2) + \partial_x(\alpha_2\rho_2 u_2^2 + \alpha_2 p_2) - p_i \partial_x \alpha_2 & = & \theta_u (u_1 - u_2). \end{cases} \quad (1)$$

The orange terms are closed according to the situation (bubbly flow, stratified flow, . . .).

[6] M.R. Baer and J.W. Nunziato. "A Two-Phase Mixture Theory for the Deflagration-to-Detonation Transition (DDT) in Reactive Granular Materials". In: *International Journal of Multiphase Flow* 12.6 (Nov. 1986), pp. 861–889.

Outline

2. Derivation of an averaged model for a stratified bifluid flow
   - Equations and boundary conditions
   - Rescaling and asymptotic analysis
   - Averaging and closing the model
   - Final averaged model

Aim of the internship

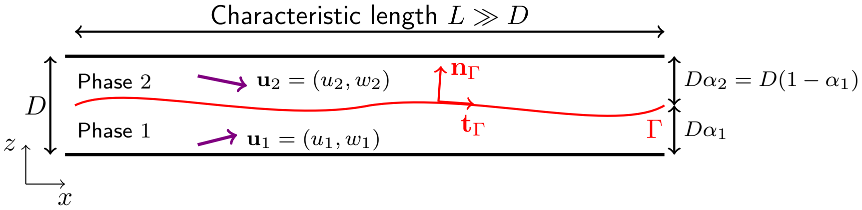

Figure: Flow configuration[7]

- Derive a Baer-Nunziato-type model for the configuration above by the dimension reduction technique.
- Study the closing of the orange terms in the model.

[7] Edwige Godlewski and Nicolas Seguin. *Modèles hyperboliques d'écoulements complexes dans le domaine de l'énergie*. 2011.

The Navier-Stokes equations

On their respective domain, the fluid flows are described by the Navier-Stokes equations

$$\partial_t \rho_k + \mathrm{div}(\rho_k \mathbf{u}_k) = 0, \tag{2}$$

$$\partial_t(\rho_k \mathbf{u}_k) + \mathrm{div}(\rho_k \mathbf{u}_k \otimes \mathbf{u}_k - \sigma_k) = 0, \tag{3}$$

with $\sigma_k = (-p_k - \mu_k \mathrm{div}(\mathbf{u}_k))\,\mathrm{Id} + \mu_k \left((\nabla \otimes \mathbf{u}_k) + {}^t(\nabla \otimes \mathbf{u}_k)\right).$

Boundary conditions

1. **At the top and bottom:**

$$\begin{cases} (\kappa_1 u_1 - \mu_1 \partial_z u_1)\big|_{z=0} = 0, \\ w_1\big|_{z=0} = 0, \end{cases} \quad \text{et} \quad \begin{cases} (\kappa_2 u_2 + \mu_2 \partial_z u_2)\big|_{z=D} = 0, \\ w_2\big|_{z=D} = 0, \end{cases} \tag{4}$$

2. **At the interface:**

$$(\sigma_1 \mathbf{n}_\Gamma)\big|_{z=D\alpha_1} = (\sigma_2 \mathbf{n}_\Gamma)\big|_{z=D\alpha_1}, \tag{5}$$

$$(\mathbf{u}_1 \cdot \mathbf{n}_\Gamma)\big|_{z=D\alpha_1} = (\mathbf{u}_2 \cdot \mathbf{n}_\Gamma)\big|_{z=D\alpha_1}, \tag{6}$$

$$((\sigma_1 \mathbf{n}_\Gamma) \cdot \mathbf{t}_\Gamma - \kappa_i(\mathbf{u}_1 \cdot \mathbf{t}_\Gamma - \mathbf{u}_2 \cdot \mathbf{t}_\Gamma))\big|_{z=D\alpha_1} = 0, \tag{7}$$

Advection of the interface

Let χ_1 denote the characteristic function of the domain occupied by the first fluid. The interface is advected by the velocity :

$$\partial_t \chi_1 + \mathbf{u}_1(\alpha_1) \cdot \nabla \chi_1 = 0.$$

This equation can be written as

$$\partial_t \alpha_1 + u_1(D\alpha_1)\partial_x \alpha_1 = \frac{w_1(D\alpha_1)}{D}. \tag{8}$$

Derivation of an averaged model

1. **Hypotheses:**
   - $\varepsilon = D/L \ll 1$ and $w_k = O(\varepsilon)$ for $k = 1, 2$.
   - Barotropic flow: $p_k = p_k(\rho_k)$.
   - μ, κ and κ_i may depend on ε.
2. **Rescaling:** the dependence on ε becomes explicit.
3. **Asymptotic analysis**
4. **Averaging:** integration of the equations on each fluid's height to get a system of 1D equations depending on the averaged unknowns.
5. **Closing the equations:** determine the boundary terms of the integration thanks to the boundary conditions.
6. **Limit** $\varepsilon \to 0$ and identification of the one dimensional model reached.

Rescaling

The rescaled Navier-Stokes equations read : $\forall k = 1, 2$,

$$\begin{aligned}
\partial_t \alpha_1 + u_1(\alpha_1)\partial_x \alpha_1 =& w_1(\alpha_1), \\
\partial_t \rho_k + \partial_x(\rho_k u_k) + \partial_z(\rho_k w_k) =& 0, \\
\partial_t(\rho_k u_k) + \partial_x(\rho_k u_k^2 + p_k) + \partial_z(\rho_k u_k w_k) =& \mu_k \partial_{xx} u_k + \frac{\mu_k}{\varepsilon^2}\partial_{zz} u_k, \\
\varepsilon^2(\partial_t(\rho_k w_k) + \partial_x(\rho_k u_k w_k) + \partial_z(\rho_k w_k^2)) + \partial_z p_k =& \mu_k \varepsilon^2 \partial_{xx} w_k + \mu_k \partial_{zz} w_k.
\end{aligned}$$

Asymptotic analysis

The Navier-Stokes system

Neglecting the terms going to 0 when $\varepsilon \to 0$, we get

$$
\begin{aligned}
\partial_t \alpha_1 + u_1^\varepsilon(\alpha_1)\partial_x \alpha_1 =& w_1^\varepsilon(\alpha_1), && (9)\\
\partial_t \rho_k^\varepsilon + \partial_x(\rho_k^\varepsilon u_k^\varepsilon) + \partial_z(\rho_k^\varepsilon w_k^\varepsilon) =& 0, && (10)\\
\partial_t(\rho_k^\varepsilon u_k^\varepsilon) + \partial_x(\rho_k^\varepsilon (u_k^\varepsilon)^2 + p_k^\varepsilon) + \partial_z(\rho_k^\varepsilon u_k^\varepsilon w_k^\varepsilon) =& \mu_k \partial_{xx} u_k^\varepsilon + \frac{\mu_k}{\varepsilon^2}\partial_{zz} u_k^\varepsilon, && (11)\\
\partial_z p_k^\varepsilon =& \mu_k \partial_{zz} w_k^\varepsilon + O(\varepsilon^2). && (12)
\end{aligned}
$$

Asymptotic analysis

The boundary conditions

The boundary conditions become

$$\left(\varepsilon\kappa_1 u_1^\varepsilon - \mu_1\partial_z u_1^\varepsilon\right)\Big|_{z=0} = 0, \quad (13)$$

$$w_1^\varepsilon\Big|_{z=0} = 0, \quad (14)$$

$$\left(\kappa_2 u_2^\varepsilon + \mu_2\partial_z u_2^\varepsilon\right)\Big|_{z=1} = 0, \quad (15)$$

$$w_2^\varepsilon\Big|_{z=1} = 0, \quad (16)$$

$$\left(\partial_x\alpha_1(p_1^\varepsilon - \mu_1(\partial_x u_1^\varepsilon - \partial_z w_1^\varepsilon)) + \mu_1\partial_x w_1^\varepsilon + \frac{\mu_1}{\varepsilon^2}\partial_z u_1^\varepsilon\right)\Big|_{z=\alpha_1} = (\dots)_2\Big|_{z=\alpha_1}, \quad (17)$$

$$\left(p_1^\varepsilon + \mu_1\partial_x\alpha_1\partial_z u_1^\varepsilon - \mu_1\partial_z w_1^\varepsilon + \mu_1\partial_x u_1^\varepsilon\right)\Big|_{z=\alpha_1} = (\dots)_2\Big|_{z=\alpha_1} + O(\mu\varepsilon^2), \quad (18)$$

$$\left(-u_1^\varepsilon\partial_x\alpha_1 + w_1^\varepsilon\right)\Big|_{z=\alpha_1} = (\cdots)_2\Big|_{z=\alpha_1}, \quad (19)$$

$$\left(\frac{\mu_1}{\varepsilon}\partial_z u_1^\varepsilon + \kappa_i(u_1^\varepsilon - u_2^\varepsilon)\right)\Big|_{z=\alpha_1} = O(\mu\varepsilon + \kappa_i\varepsilon^2), \quad (20)$$

Averaging

Principle

- Integration of the Navier-Stokes equations with respect to z.
- Two types of averages : $\overline{f} = \frac{1}{\alpha_k} \int f(z) \mathrm{d}z$ and $\widehat{f} = \frac{\overline{\rho f}}{\overline{\rho}}$.
- New unknown : the effective flux $F_k^\varepsilon = p_k^\varepsilon - \mu \partial_x u_k^\varepsilon$.
- Presence of boundary terms in the equations that must be closed in order to get a model depending only on the averaged unknowns (equation (9) and boundary conditions).

Averaging

Averaged Navier-Stokes

$$\partial_t(\alpha_1 \overline{\rho_1^\varepsilon}) + \partial_x(\alpha_1 \overline{\rho_1^\varepsilon}\,\widehat{u_1^\varepsilon}) = 0, \tag{21}$$

$$\partial_t(\alpha_2 \overline{\rho_2^\varepsilon}) + \partial_x(\alpha_2 \overline{\rho_2^\varepsilon}\,\widehat{u_2^\varepsilon}) = 0, \tag{22}$$

$$\begin{aligned}\partial_t(\alpha_1 \overline{\rho_1^\varepsilon}\,\widehat{u_1^\varepsilon}) + \partial_x(\alpha_1(\overline{\rho_1^\varepsilon}\,\widehat{u_1^\varepsilon}^2 + \overline{F_1^\varepsilon})) =& -\frac{\kappa_1}{\varepsilon} u_1^\varepsilon(0) + \frac{\mu_1}{\varepsilon^2}\partial_z u_1^\varepsilon(\alpha_1) \\ &+ F_1^\varepsilon(\alpha_1)\partial_x \alpha_1 + O\left(\frac{\varepsilon^2\kappa^2 + \varepsilon^4}{\mu^2}\right),\end{aligned} \tag{23}$$

$$\begin{aligned}\partial_t(\alpha_2 \overline{\rho_2^\varepsilon}\,\widehat{u_2^\varepsilon}) + \partial_x(\alpha_2(\overline{\rho_2^\varepsilon}\,\widehat{u_2^\varepsilon}^2 + \overline{F_2^\varepsilon})) =& -\frac{\kappa_2}{\varepsilon} u_2^\varepsilon(1) - \frac{\mu_2}{\varepsilon^2}\partial_z u_2^\varepsilon(\alpha_1) \\ &+ F_2^\varepsilon(\alpha_1)\partial_x \alpha_2 + O\left(\frac{\varepsilon^2\kappa^2 + \varepsilon^4}{\mu^2}\right).\end{aligned} \tag{24}$$

Closing of the friction terms at the boundaries of Ω

We use the approximation

$$\begin{cases} u_1^\varepsilon(0) &= \widehat{u_1^\varepsilon} + O\left(\frac{\varepsilon\kappa_k+\varepsilon^2}{\mu_k}\right), \\ u_2^\varepsilon(1) &= \overline{u_2^\varepsilon} + O\left(\frac{\varepsilon\kappa_k+\varepsilon^2}{\mu_k}\right). \end{cases}$$

Reminder of the equations

$$
\begin{aligned}
\partial_t(\alpha_1\overline{\rho_1^\varepsilon}) + \partial_x(\alpha_1\overline{\rho_1^\varepsilon}\,\widehat{u_1^\varepsilon}) =& 0,\\
\partial_t(\alpha_2\overline{\rho_2^\varepsilon}) + \partial_x(\alpha_2\overline{\rho_2^\varepsilon}\,\widehat{u_2^\varepsilon}) =& 0,\\
\partial_t(\alpha_1\overline{\rho_1^\varepsilon}\,\widehat{u_1^\varepsilon}) + \partial_x(\alpha_1(\overline{\rho_1^\varepsilon}\,\widehat{u_1^\varepsilon}^2 + \overline{F_1^\varepsilon})) =& -\frac{\kappa_1}{\varepsilon}\widehat{u_1^\varepsilon} + \frac{\mu_1}{\varepsilon^2}\partial_z u_1^\varepsilon(\alpha_1)\\
&+ F_1^\varepsilon(\alpha_1)\partial_x\alpha_1\\
&+ O\left(\frac{\kappa_k^2+\varepsilon\kappa_k}{\mu_k} + \frac{\varepsilon^2\kappa_k^2+\varepsilon^4}{\mu_k^2}\right),\\
\partial_t(\alpha_2\overline{\rho_2^\varepsilon}\,\widehat{u_2^\varepsilon}) + \partial_x(\alpha_2(\overline{\rho_2^\varepsilon}\,\widehat{u_2^\varepsilon}^2 + \overline{F_2^\varepsilon})) =& -\frac{\kappa_2}{\varepsilon}\widehat{u_2}^\varepsilon - \frac{\mu_2}{\varepsilon^2}\partial_z u_2^\varepsilon(\alpha_1)\\
&+ F_2^\varepsilon(\alpha_1)\partial_x\alpha_2\\
&+ O\left(\frac{\kappa_k^2+\varepsilon\kappa_k}{\mu_k} + \frac{\varepsilon^2\kappa_k^2+\varepsilon^4}{\mu_k^2}\right).
\end{aligned}
$$

Closing of the friction terms at the interface

Using the Navier condition (20),

$$\frac{\mu_1}{\varepsilon^2}\partial_z u_1^\varepsilon(\alpha_1) = -\frac{\kappa_i}{\varepsilon}(\widehat{u_1^\varepsilon} - \widehat{u_2^\varepsilon}) + O\left(\mu + \frac{\kappa\kappa_i + \varepsilon\kappa_i}{\mu}\right).$$

Reminder of the equations

$$
\begin{aligned}
\partial_t(\alpha_1\overline{\rho_1^\varepsilon}) + \partial_x(\alpha_1\overline{\rho_1^\varepsilon}\,\widehat{u_1^\varepsilon}) =& 0,\\
\partial_t(\alpha_2\overline{\rho_2^\varepsilon}) + \partial_x(\alpha_2\overline{\rho_2^\varepsilon}\,\widehat{u_2^\varepsilon}) =& 0,\\
\partial_t(\alpha_1\overline{\rho_1^\varepsilon}\,\widehat{u_1^\varepsilon}) + \partial_x(\alpha_1(\overline{\rho_1^\varepsilon}\,\widehat{u_1^\varepsilon}^2 + \overline{F_1^\varepsilon})) =& -\frac{\kappa_1}{\varepsilon}\widehat{u_1^\varepsilon} - \frac{\kappa_i}{\varepsilon}(\widehat{u_1^\varepsilon} - \widehat{u_2^\varepsilon})\\
&+ F_1^\varepsilon(\alpha_1)\partial_x\alpha_1 + R,\\
\partial_t(\alpha_2\overline{\rho_2^\varepsilon}\,\widehat{u_2^\varepsilon}) + \partial_x(\alpha_2(\overline{\rho_2^\varepsilon}\,\widehat{u_2^\varepsilon}^2 + \overline{F_2^\varepsilon})) =& -\frac{\kappa_2}{\varepsilon}\widehat{u_2}^\varepsilon + \frac{\kappa_i}{\varepsilon}(\widehat{u_1^\varepsilon} - \widehat{u_2^\varepsilon})\\
&+ F_2^\varepsilon(\alpha_1)\partial_x\alpha_2 + R.
\end{aligned}
$$

where

$$
R = O\left(\mu + \frac{\kappa_i(\kappa+\varepsilon) + \kappa^2 + \kappa\varepsilon}{\mu} + \frac{\varepsilon^2\kappa^2 + \varepsilon^4}{\mu^2}\right).
$$

Simplifying the model

Discussing orders of magnitude

- **About the pressures :** $p_1^\varepsilon(\alpha_1) = p_2^\varepsilon(\alpha_1) + O(\mu)$, $F_k^\varepsilon = p_k^\varepsilon + O(\mu)$ and $\overline{p_1^\varepsilon} = \overline{p_2^\varepsilon} + O(\mu)$.

Simplifying the model

Discussing orders of magnitude

- **About the pressures :** $p_1^\varepsilon(\alpha_1) = p_2^\varepsilon(\alpha_1) + O(\mu)$, $F_k^\varepsilon = p_k^\varepsilon + O(\mu)$ and $\overline{p_1^\varepsilon} = \overline{p_2^\varepsilon} + O(\mu)$.
- **About the viscosities and friction coefficients :**

$$\mu = \underline{\mu}\varepsilon^\tau,\ 0 < \tau \leqslant 2, \quad \kappa = \underline{\kappa}\varepsilon^\xi,\ \xi > 1 \text{ and } \kappa_i = \underline{\kappa_i}\varepsilon.$$

Final averaged model

We take the case where $\xi > 1$. Then,

$$\partial_t(\alpha_1\overline{\rho_1}) + \partial_x(\alpha_1\overline{\rho_1}\,\widehat{u_1}) = 0, \tag{25}$$
$$\partial_t(\alpha_2\overline{\rho_2}) + \partial_x(\alpha_2\overline{\rho_2}\,\widehat{u_2}) = 0, \tag{26}$$
$$\partial_t(\alpha_1\overline{\rho_1}\,\widehat{u_1}) + \partial_x(\alpha_1\overline{\rho_1}\,\widehat{u_1}^2 + \alpha_1\overline{p_1}) - p_i\partial_x\alpha_1 = -\frac{\kappa_i}{2}(\widehat{u_1} - \widehat{u_2}), \tag{27}$$
$$\partial_t(\alpha_2\overline{\rho_2}\,\widehat{u_2}) + \partial_x(\alpha_2\overline{\rho_2}\,\widehat{u_2}^2 + \alpha_2\overline{p_2}) - p_i\partial_x\alpha_2 = \frac{\kappa_i}{2}(\widehat{u_1} - \widehat{u_2}), \tag{28}$$
$$\alpha_1 + \alpha_2 = 1, \tag{29}$$
$$\overline{p_1} = p_1(\overline{\rho_1}), \tag{30}$$
$$\overline{p_2} = p_2(\overline{\rho_2}), \tag{31}$$
$$p_1(\overline{\rho_1}) = p_2(\overline{\rho_2}) = p_i. \tag{32}$$

Reminder of the Baer-Nunziato mode

$$\begin{cases} \partial_t \alpha_1 + v_i \partial_x \alpha_1 & = & \theta_p (p_1 - p_2), \\ \partial_t (\alpha_1 \rho_1) + \partial_x (\alpha_1 \rho_1 u_1) & = & 0, \\ \partial_t (\alpha_1 \rho_1 u_1) + \partial_x (\alpha_1 \rho_1 u_1^2 + \alpha_1 p_1) - p_i \partial_x \alpha_1 & = & \theta_u (u_2 - u_1), \\ \partial_t (\alpha_2 \rho_2) + \partial_x (\alpha_2 \rho_2 u_2) & = & 0, \\ \partial_t (\alpha_2 \rho_2 u_2) + \partial_x (\alpha_2 \rho_2 u_2^2 + \alpha_2 p_2) - p_i \partial_x \alpha_2 & = & \theta_u (u_1 - u_2). \end{cases}$$

- No advection equation for the interface in our model.
- We get a single-pressure model instead of getting two distinct pressures.

Conclusion and perspectives

About the model:

- Obtention of the isentropic version of the standard model for bifluid flows described in [DP99; IH11], called the two-velocities, one-pressure model.
- This model is non-hyperbolic, but it can be stabilized with some corrective terms.[8]

[8] D. Bresch et al. "Global Weak Solutions to a Generic Two-Fluid Model". In: *Archive for Rational Mechanics and Analysis* 196.2 (May 2010), pp. 599–629.

Conclusion and perspectives

About the model:

- Obtention of the isentropic version of the standard model for bifluid flows described in [DP99; IH11], called the two-velocities, one-pressure model.
- This model is non-hyperbolic, but it can be stabilized with some corrective terms.[8]

Perspectives:

- By taking $\underline{\kappa_i} \to +\infty$, we obtain a model with one velocity and one pressure, which is strictly hyperbolic.
- Prove the convergence of solutions to the initial Navier-Stokes equations for the bifluid flow to a solution to the one-velocity, one-pressure model using a relative entropy procedure.

[8] D. Bresch et al. "Global Weak Solutions to a Generic Two-Fluid Model". In: *Archive for Rational Mechanics and Analysis* 196.2 (May 2010), pp. 599–629.

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[GS11] Edwige Godlewski and Nicolas Seguin. *Modèles hyperboliques d'écoulements complexes dans le domaine de l'énergie*. 2011.

[Bre+10] D. Bresch et al. "Global Weak Solutions to a Generic Two-Fluid Model". In: *Archive for Rational Mechanics and Analysis* 196.2 (May 2010), pp. 599–629.