Derivation of averaged compressible two-phase flow models

Pierrick Le Vourc'h

Institut Montpelliérain Alexander Grothendieck and Inria ANGUS team
Université de Montpellier

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Outline

What are we talking about?

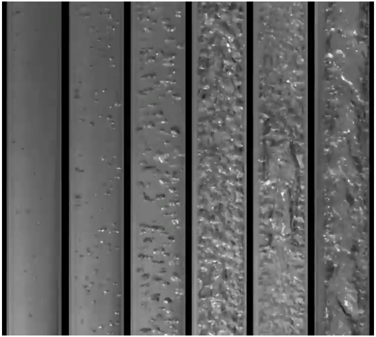

Figure: Different regimes of bubbly flows.
Source : Zorbubbles video of the American Physical Society's YouTube channel

Figure: Stratified flow.
Source : Video by National Committee of Fluid Mechanics

About two-phase flow models

Why?

- Navier-Stokes mathematically unsolvable.
- Numerical simulation too expensive.
- Navier-Stokes is too precise.[1]

[1] Drew and Passman. *Theory of Multicomponent Fluids*. Springer New York, 1999

[2] Ishii and Hibiki. *Thermo-Fluid Dynamics of Two-Phase Flow*. Springer, 2011

[3] Stewart and Wendroff. "Two-phase flow: Models and methods". In: *Journal of Computational Physics* (1984)

[4] Bresch, Burtea, and Lagoutière. "Mathematical Justification of a Compressible Bifluid System with Different Pressure Laws: a Continuous Approach". In: *Applicable Analysis* (2022)

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How?

- Apply averaging operators.
- Different types of averaging: time, space.[2] [3]
- Homogenization.[4]

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[2] Ishii and Hibiki. *Thermo-Fluid Dynamics of Two-Phase Flow*. Springer, 2011

[3] Stewart and Wendroff. "Two-phase flow: Models and methods". In: *Journal of Computational Physics* (1984)

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Different types of models

- **One velocity, one pressure:** well-posed on a finite time interval for regular initial conditions.
- **Two velocities, one pressure:** ill-posed. Can be stabilized with additional terms like viscosity, surface tension, or capillarity.[5] [6]
- **One velocity, two pressures:** well-posed on a finite time interval for regular initial conditions. Obtained in 1D by homogenization.
- **Two velocities, two pressures :** well-posed on a finite time interval for regular initial conditions.[7] [8]

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[5] Ramshaw and Trapp. "Characteristics, Stability, and Short-Wavelength Phenomena in Two-Phase Flow Equation Systems". In: *Nuclear Science and Engineering* (1978)

[6] Bresch et al. "Global Weak Solutions to a Generic Two-Fluid Model". In: *Archive for Rational Mechanics and Analysis* (2010)

[7] Baer and Nunziato. "A Two-Phase Mixture Theory for the Deflagration-to-Detonation Transition (DDT) in Reactive Granular Materials". In: *International Journal of Multiphase Flow* (1986)

[8] Coquel et al. "Two Properties of Two-Velocity Two-Pressure Models for Two-Phase Flows". In: *Communications in Mathematical Sciences* (2014)

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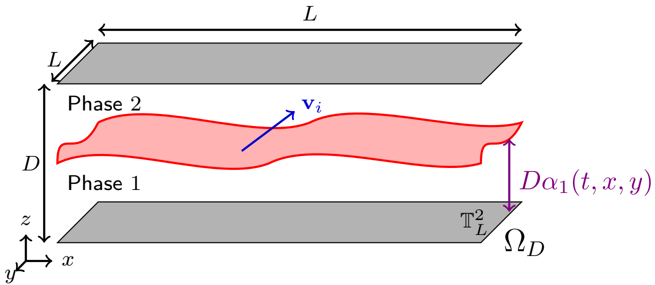
Flow configuration



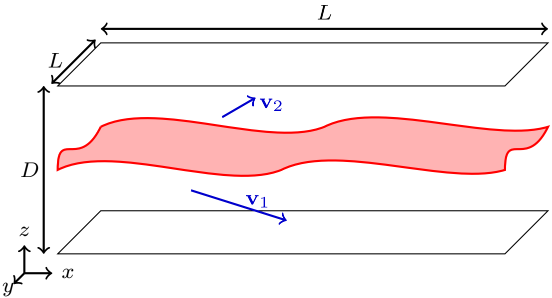
Assumptions

- The fluids are immiscible.
- The interface is decribed by a continuous function $\alpha_1 \Rightarrow$ the stratification is preserved through time.

Kinematic equation for the interface

$$\partial_t(D\alpha_1) - \mathbf{v}_i \cdot \begin{pmatrix} -\nabla_h(D\alpha_1) \\ 1 \end{pmatrix} = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{T}_L^2.$$

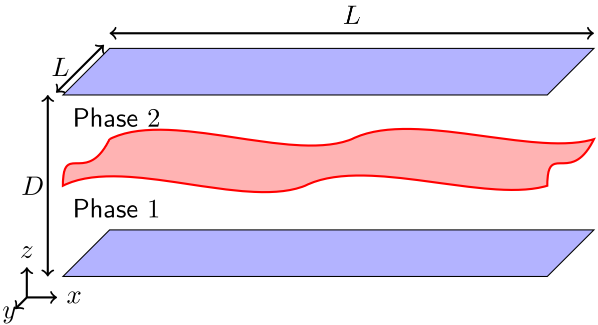
The barotropic compressible Navier-Stokes equations



$$\begin{cases} \partial_t \rho_k + \nabla \cdot (\rho_k \mathbf{v}_k) &= 0, \\ \partial_t(\rho_k \mathbf{v}_k) + \nabla \cdot (\rho_k \mathbf{v}_k \otimes \mathbf{v}_k) + \nabla p_k &= \nabla \cdot \mathbb{S}_k, \\ p_k &= p_k(\rho_k), \end{cases}$$

where
$$\mathbb{S}_k = 2\mu_k \frac{(\nabla \otimes \mathbf{v}_k + (\nabla \otimes \mathbf{v}_k)^T)}{2} + \lambda_k (\nabla \cdot \mathbf{v}_k)\mathbb{I}$$

Boundary conditions



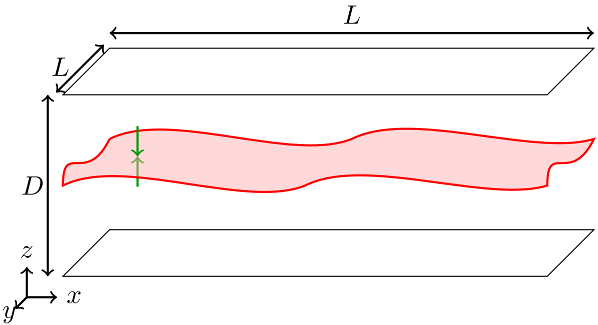
Bottom

$$\begin{cases} w_1\big|_{z=0} = & 0, \\ -\mathbb{S}_{1,hz} + \kappa_1 \mathbf{v}_{1,h}\big|_{z=0} = & 0, \end{cases}$$

Top

$$\begin{cases} w_2\big|_{z=D} = & 0, \\ \mathbb{S}_{2,hz} + \kappa_2 \mathbf{v}_{2,h}\big|_{z=D} = & 0. \end{cases}$$

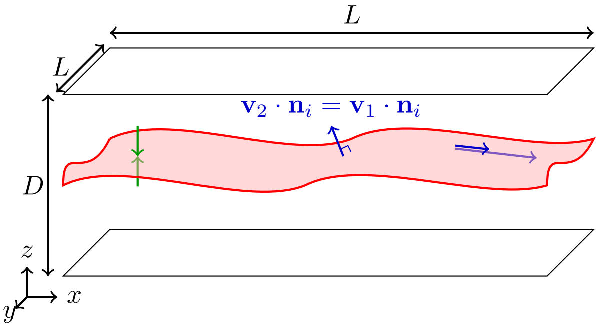
Interface conditions



Continuity of the normal stress

$$(-p_1\mathbb{I} + \mathbb{S}_1)\mathbf{n}_i|_{z=D\alpha_1} = (-p_2\mathbb{I} + \mathbb{S}_2)\mathbf{n}_i|_{z=D\alpha_1}.$$

Interface conditions



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Navier friction condition

$$\begin{cases} \forall k \in \{1,2\}, \qquad (\mathbf{v}_k - \mathbf{v}_i)\cdot \mathbf{n}_i|_{z=D\alpha_1} = 0, \\ \qquad (\mathbb{S}_1\mathbf{n}_i + \kappa_i(\mathbf{v}_1 - \mathbf{v}_2))_{\tan}|_{z=D\alpha_1} = 0. \end{cases}$$

Outline

The asymptotic study

Assumptions

- Thickness parameter $\varepsilon = D/L \ll 1$.
- Small vertical velocity : $w = O(\varepsilon)$.
- The viscosity and friction coefficients might depend on ε.

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- Small vertical velocity : $w = O(\varepsilon)$.
- The viscosity and friction coefficients might depend on ε.

Consequence

- The velocities and the densities don't vary much in the vertical direction.

Hope

Finding a model written in terms of vertical averages might be relevant.

Derivation protocol

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2. Neglect terms of high order in ε.
3. Integrate the two Navier-Stokes systems in the vertical direction.

Definition (Averages)

$$\overline{\rho_k}(t,x,y) = \frac{1}{\alpha_k}\int_{H_k} \rho_k(t,x,y,z)\mathrm{d}z.$$

$$\langle \mathbf{v}_{k,h}\rangle(t,x,y) = \frac{\overline{\rho_k \mathbf{v}_{k,h}}}{\overline{\rho_k}} = \frac{1}{\alpha_k\overline{\rho_k}}\int_{H_k} \rho_k \mathbf{v}_k(t,x,y,z)\mathrm{d}z.$$

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4. Close the remainining boundary terms using the boundary conditions and estimates.

The two-velocity, one-pressure model

Scaling the viscosity and friction coefficients

$$\forall k \in \{1,2\}, \qquad \begin{cases} \mu_k = \widehat{\mu_k}\varepsilon^\tau, \quad \lambda_k = \widehat{\lambda_k}\varepsilon^\tau, \quad 0 < \tau < 2; \\ \kappa_i = \widehat{\kappa_i}\varepsilon, \quad \kappa_k = \widehat{\kappa_k}\varepsilon. \end{cases} \Longrightarrow \text{one pressure}$$

Theorem (Averaged model)

$$\partial_t(\alpha_k\overline{\rho_k}) + \nabla_h\cdot(\alpha_k\overline{\rho_k}\langle\mathbf{v}_{k,h}\rangle) = 0,$$

$$\partial_t(\alpha_k\overline{\rho_k}\langle\mathbf{v}_{k,h}\rangle) + \nabla_h\cdot(\alpha_k\overline{\rho_k}\langle\mathbf{v}_{k,h}\rangle\otimes\langle\mathbf{v}_{k,h}\rangle) + \nabla_h(\alpha_k\overline{p_k}) - \overline{p_k}\nabla_h\alpha_k$$
$$= -\widehat{\kappa_k}\langle\mathbf{v}_{k,h}\rangle + (-1)^k\widehat{\kappa_i}(\langle\mathbf{v}_{1,h}\rangle - \langle\mathbf{v}_{2,h}\rangle),$$

$$\overline{p_1} = p_1(\overline{\rho_1}) = p_2(\overline{\rho_2}) = \overline{p_2},$$

$$\alpha_1 + \alpha_2 = 1.$$

The momentum equations are satisfied up to an error of order $\min(\tau, 2-\tau)$*, the barotropy equation is satisfied up to an error of order* τ*. The others are exact.*

What else ?

In addition to being able to derive the two-velocity, one pressure model, our protocol can be used to derive:

1. The two-velocity, one-pressure model with a quadratic drag term.
2. The one-velocity, one-pressure model.
3. The two-velocity, one-pressure and two-temperature model in the Navier-Stokes Fourier case.
4. And others . . .

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Take-home message

Design of a simple protocol to derive a variety of averaged models for stratified compressible flows in thin domains.

Outline

Conclusion and perspectives

Conclusion

- Derivation of the two-velocity, one-pressure model without heuristic assumptions.
- The protocol used in this derivation can be adapted and extended to work with other models.

Perspectives

- Study the well-posedness of the microscopic model.
- Work on a multilayer two-phase flow model.

Thank you for your attention!

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