

Formal derivation of a stratified compressible two-phase flow model

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Outline

- 1 Averaged two-phase flow models
- 2 Derivation of a two-velocity, one-pressure model

What are we talking about?

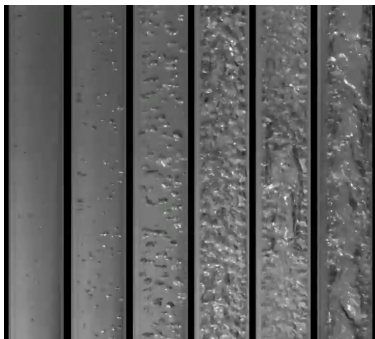


Figure: Different regimes of bubbly flows.

Source : [Zorbubbles video of the American Physical Society's YouTube channel](#)



Figure: Stratified flow. Source : [Video by National Committee of Fluid Mechanics](#)

About two-phase flow models

Why?

- Navier-Stokes systems are complex.
- Numerical simulation too expensive.
- Need of repeatability.¹

¹Drew and Passman. *Theory of Multicomponent Fluids*. Springer New York, 1999

²Ishii and Hibiki. *Thermo-Fluid Dynamics of Two-Phase Flow*. Springer, 2011

³Stewart and Wendroff. "Two-phase flow: Models and methods". In: *Journal of Computational Physics* (1984)

⁴Bresch, Burtea, and Lagoutière. "Mathematical Justification of a Compressible Bifluid System with Different Pressure Laws: a Continuous Approach". In: *Applicable Analysis* (2022)

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How?

- Apply averaging operators.
- Different types of averaging: time, space ...^{2 3}
- Homogenization.⁴

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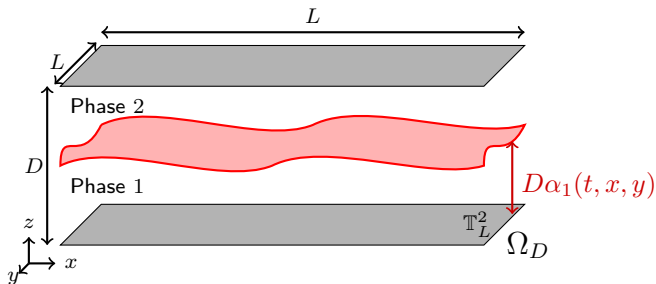
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Flow configuration⁵

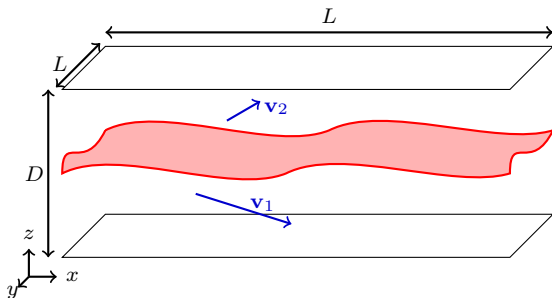


Assumptions

- The fluids are immiscible.
- The interface is described by a continuous function $\alpha_1 \Rightarrow$ the stratification is preserved through time.

⁵Pierrick Le Vourc'h, Khaled Saleh, and Nicolas Seguin. "Averaged models for compressible two-phase stratified flows on thin domains". Preprint. June 2025. URL: <https://hal.science/hal-05096837>

The barotropic compressible Navier-Stokes equations

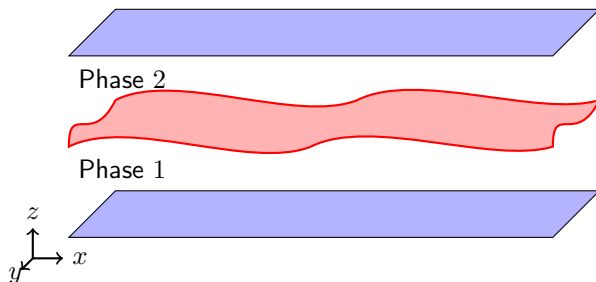


$$\begin{cases} \partial_t \rho_k + \nabla \cdot (\rho_k \mathbf{v}_k) &= 0, \\ \partial_t (\rho_k \mathbf{v}_k) + \nabla \cdot (\rho_k \mathbf{v}_k \otimes \mathbf{v}_k) + \nabla p_k &= \nabla \cdot \mathbb{S}_k, \\ p_k &= p_k(\rho_k), \end{cases}$$

where

$$\mathbb{S}_k = 2\mu_k \frac{(\nabla \otimes \mathbf{v}_k + (\nabla \otimes \mathbf{v}_k)^T)}{2} + \lambda_k (\nabla \cdot \mathbf{v}_k) \mathbb{I}$$

Boundary conditions



Bottom

$$\begin{cases} w_1|_{z=0} = 0, \\ -\mu_1 \partial_z \mathbf{v}_{1,h} + \kappa_1 \mathbf{v}_{1,h}|_{z=0} = 0, \end{cases}$$

Top

$$\begin{cases} w_2|_{z=D} = 0, \\ \mu_2 \partial_z \mathbf{v}_{2,h} + \kappa_2 \mathbf{v}_{2,h}|_{z=D} = 0. \end{cases}$$

Interface dynamics



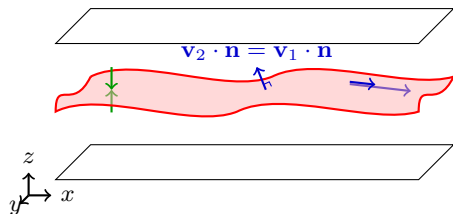
Continuity of the normal stress

$$[(-p\mathbb{I} + \mathbb{S})\mathbf{n}]_{z=D\alpha_1} = 0.$$

Continuity of the normal velocity

$$[\mathbf{v} \cdot \mathbf{n}]_{z=D\alpha_1} = 0.$$

Interface dynamics



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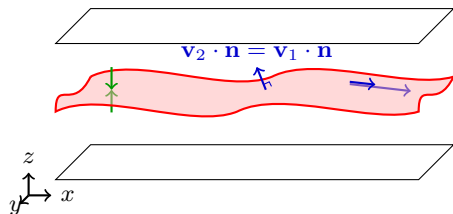
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$$[\mathbf{v} \cdot \mathbf{n}]_{z=D\alpha_1} = 0.$$

Tangential friction

$$(\mathbb{S}_1\mathbf{n} - \kappa_i[\mathbf{v}])_{\text{tan}}|_{z=D\alpha_1} = 0.$$

Interface dynamics



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Tangential friction

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Kinematic equation for the interface

$$\partial_t(D\alpha_1) - \mathbf{v}_1(D\alpha_1) \cdot \left(\frac{-\nabla_h(D\alpha_1)}{1} \right) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{T}_L^2.$$

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The asymptotic study

Assumptions

- Thickness parameter $\varepsilon = D/L \ll 1$.
- Small vertical velocity : $w = O(\varepsilon)$.
- The viscosity and friction coefficients might depend on ε .

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- Small vertical velocity : $w = O(\varepsilon)$.
- The viscosity and friction coefficients might depend on ε .

Consequence

- The velocities and the densities don't vary much in the vertical direction.

Hope

Finding a model written in terms of vertical averages might be relevant.

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- 3 Integrate the two Navier-Stokes systems in the vertical direction.

Definition (Averages)

$$\overline{\rho_k}(t, x, y) = \frac{1}{\alpha_k} \int \rho_k(t, x, y, z) dz.$$

$$\langle \mathbf{v}_{k,h} \rangle(t, x, y) = \frac{\overline{\rho_k \mathbf{v}_{k,h}}}{\overline{\rho_k}} = \frac{1}{\alpha_k \overline{\rho_k}} \int \rho_k \mathbf{v}_k(t, x, y, z) dz.$$

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- 4 Close the remaining boundary terms using the boundary conditions and estimates.
- 4 Discuss the validity of the model at the limit $\varepsilon \rightarrow 0$.

The two-velocity, one-pressure model

Scaling the viscosity and friction coefficients

$$\forall k \in \{1, 2\}, \quad \left\{ \begin{array}{l} \mu_k = \widehat{\mu}_k \varepsilon^\tau, \quad \lambda_k = \widehat{\lambda}_k \varepsilon^\tau, \quad 0 < \tau < 2; \\ \kappa_i = \widehat{\kappa}_i \varepsilon, \quad \kappa_k = \widehat{\kappa}_k \varepsilon. \end{array} \right. \implies \text{one pressure}$$

Theorem (Averaged model)

$$\begin{aligned} \partial_t(\alpha_k \overline{\rho_k}) + \nabla_h \cdot (\alpha_k \overline{\rho_k} \langle \mathbf{v}_{k,h} \rangle) &= 0, \\ \partial_t(\alpha_k \overline{\rho_k} \langle \mathbf{v}_{k,h} \rangle) + \nabla_h \cdot (\alpha_k \overline{\rho_k} \langle \mathbf{v}_{k,h} \rangle \otimes \langle \mathbf{v}_{k,h} \rangle) + \nabla_h(\alpha_k \overline{p_k}) - \overline{p_k} \nabla_h \alpha_k \\ &= -\widehat{\kappa}_k \langle \mathbf{v}_{k,h} \rangle + (-1)^k \widehat{\kappa}_i (\langle \mathbf{v}_{1,h} \rangle - \langle \mathbf{v}_{2,h} \rangle), \\ \overline{p_1} = p_1(\overline{\rho_1}) = p_2(\overline{\rho_2}) &= \overline{p_2}, \\ \alpha_1 + \alpha_2 &= 1. \end{aligned}$$

The momentum equations are satisfied up to an error of order $\varepsilon^\tau + \varepsilon^{2-\tau}$, the barotropy equation is satisfied up to an error of order ε^τ . The others are exact.

Concluding remarks

About the two-velocity, one-pressure model

- Ill-posed model (complex eigenvalues).
- Used in the industry with stabilizing terms (viscosity, surface tension, capillarity ...^{6 7}).

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⁶Bresch et al. "Global Weak Solutions to a Generic Two-Fluid Model". In: *Archive for Rational Mechanics and Analysis* (2010)

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Our protocol can be used to derive other models:

- 1 **One velocity, one pressure:** well-posed on a finite time interval for regular initial conditions.
- 2 **Two velocities, one pressure, two temperatures**
- 3 **And others ...**

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Conclusion

- Derivation of the two-velocity, one-pressure model without heuristic assumptions.
- The protocol used in this derivation can be adapted and extended to work with other models.

Take-home message

Design of a simple protocol to derive a variety of averaged models for stratified compressible flows in thin domains.

Thank you for your attention!

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