

Mathematical analysis and numerical simulation of a Kapila model

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Outline

1 The model: presentation and context

2 Analysis of the model

3 Numerical temporary results

Goals

This work studies the Kapila model for compressible two-phase flows. The model has the following properties:

- It is **barotropic**.
- Both phases share a **common velocity field**.
- The system is at **pressure equilibrium**.
- The volume fractions are determined thanks to an **algebraic closure**.

This model has been derived in [BCT23; LSS26].

Cosmin Burtea, Timothée Crin-Barat, and Jin Tan. “Pressure-relaxation limit for a one-velocity Baer–Nunziato model to a Kapila model”. In: *Mathematical Models and Methods in Applied Sciences* (2023)

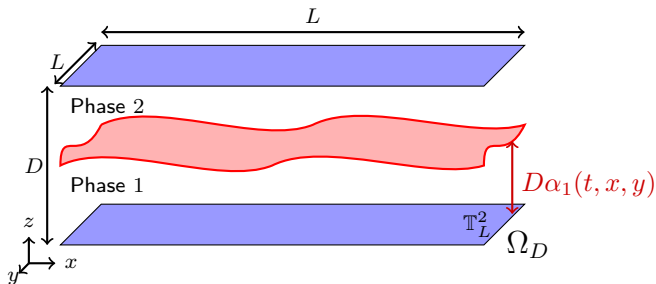
Pierrick Le Vourc’h, Khaled Saleh, and Nicolas Seguin. “Averaged models for compressible two-phase stratified flows on thin domains”. 2026

Context: pressure relaxation in the Baer-Nunziato model

- Starting point: one-velocity Baer-Nunziato
- The equation on α has a pressure relaxation term.
- Simultaneous inviscid limit and pressure relaxation.
- Convergence of the solutions with a convergence rate $\sqrt{\text{viscosity}}$.

Cosmin Burtea, Timothée Crin-Barat, and Jin Tan. "Pressure-relaxation limit for a one-velocity Baer-Nunziato model to a Kapila model". In: *Mathematical Models and Methods in Applied Sciences* (2023)

Context: formal derivation from a two-layer flow



- Two barotropic compressible Navier-Stokes systems
- Thin domain: $\varepsilon = D/L \rightarrow 0$
- Navier slip boundary conditions
- Dirichlet interface conditions + continuous stress

Depth averaging + thin layer limit + inviscid limit $\Rightarrow$ one-velocity, one-pressure two-dimensional model.

Pierrick Le Vourc'h, Khaled Saleh, and Nicolas Seguin. "Averaged models for compressible two-phase stratified flows on thin domains". 2026

The Kapila model

Notations: $\rho = \alpha_1 \rho_1 + \alpha_2 \rho_2$ total density, $y = \frac{\alpha_1 \rho_1}{\rho}$ mass fraction of phase 1.

$$\partial_t y + u \partial_x y = 0, \quad (1a)$$

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad (1b)$$

$$\partial_t(\rho u) + u \partial_x(\rho u) + \partial_x p(\rho, y) = 0, \quad (1c)$$

$$p(\rho, y) := p_1 \left(\frac{\rho y}{\alpha_1} \right) = p_2 \left(\frac{\rho(1-y)}{\alpha_2} \right), \quad (1d)$$

$$\alpha_1 + \alpha_2 = 1 \quad (1e)$$

Equations of state: positive, increasing and strictly convex.

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Hyperbolicity

Rewrite (1) as $\partial_t W + A(W) \partial_x W = 0$, (2)

where $W = \begin{pmatrix} y \\ \rho \\ u \end{pmatrix}$ and $A(W) = \begin{pmatrix} u & 0 & 0 \\ 0 & u & \rho \\ \frac{1}{\rho} \partial_y p & \frac{1}{\rho} \partial_\rho p & u \end{pmatrix}$.

¹Donald A. Drew and Stephen L. Passman. *Theory of Multicomponent Fluids*. 1999

²H. Bruce Stewart and Burton Wendroff. "Two-phase flow: Models and methods". In: *Journal of Computational Physics* (1984)

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Proposition

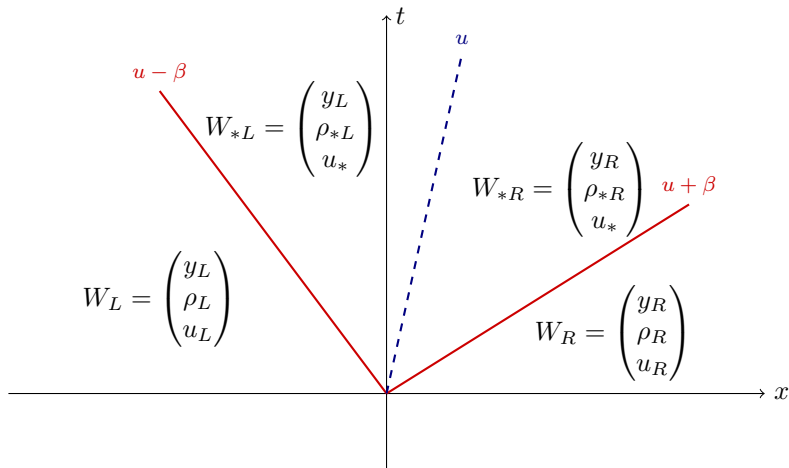
Let $\beta(\rho, y) = \sqrt{\partial_\rho p} = \frac{c_1 c_2 \sqrt{y(1-y)}}{\sqrt{\alpha_1^2 c_2^2 (1-y) + \alpha_2^2 c_1^2 y}}$ the *Wood sound speed*¹².

- ❶ System (1) is *strictly hyperbolic* with eigenvalues $\lambda_1 = u - \beta$, $\lambda_2 = u$, $\lambda_3 = u + \beta$.
- ❷ The fields associated with the eigenvalues λ_1 and λ_3 are *genuinely nonlinear*, whereas the field associated with the eigenvalue λ_2 is *linearly degenerate*.
- ❸ System (1) is *symmetrizable*.

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The Riemann problem



The Riemann invariants

- **First and third wave:** y and $u \mp \int \frac{\beta(s,y)}{s} ds$.
- **Second wave:** p and u .

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Intermediate velocities:

$$u_{*L} = \begin{cases} u_L - \sqrt{\frac{(\rho_{*L} - \rho_L)(p_{*L} - p_L)}{\rho_{*L} \rho_L}}, & \rho_{*L} > \rho_L, \\ u_L - \phi(\rho_{*L}), & \rho_{*L} \leq \rho_L, \end{cases}$$
$$u_{*R} = \begin{cases} u_R + \sqrt{\frac{(\rho_{*R} - \rho_R)(p_{*R} - p_R)}{\rho_{*R} \rho_R}}, & \rho_{*R} > \rho_R, \\ u_R + \varphi(\rho_{*R}), & \rho_{*R} \leq \rho_R. \end{cases}$$

with

$$\phi(\rho) = \int_{\rho_L}^{\rho} \frac{\beta(s,y)}{s} ds \quad \text{and} \quad \varphi(\rho) = \int_{\rho_R}^{\rho} \frac{\beta(s,y)}{s} ds$$

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Change of variables

Denote $\tilde{\rho}_k = \alpha_k \rho_k$ and $\rho = \tilde{\rho}_1 + \tilde{\rho}_2$. System (1) is equivalent to

$$\partial_t(\tilde{\rho}_1) + \partial_x(\tilde{\rho}_1 u) = 0, \quad (3a)$$

$$\partial_t(\tilde{\rho}_2) + \partial_x(\tilde{\rho}_2 u) = 0 \quad (3b)$$

$$\partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \quad (3c)$$

$$p_1 \left(\frac{\tilde{\rho}_1}{\alpha_1} \right) = p_2 \left(\frac{\tilde{\rho}_2}{\alpha_2} \right), \quad (3d)$$

$$\alpha_1 + \alpha_2 = 1. \quad (3e)$$

Pressure equation of state:

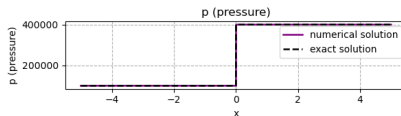
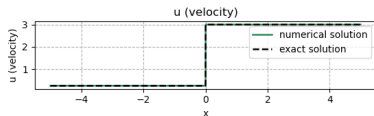
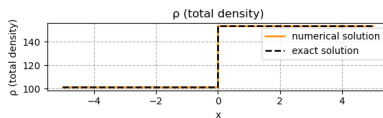
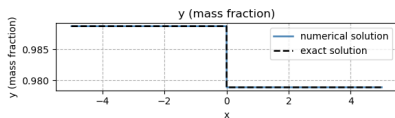
$$p_k(\rho_k) = p_0 + c_k^2(\rho_k - \rho_{0,k}).$$

The numerical scheme

- Finite volume scheme in 1D.
- Explicit Euler method with Δt updated at each iteration.
- Split computations:
 - 1 Update the phasic densities, the velocity and the pressure with FV scheme;
 - 2 Compute the volume fraction by solving $p_1 - p_2 = 0$.

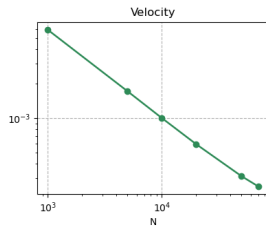
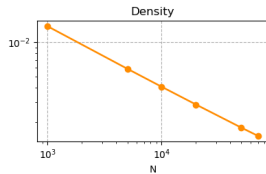
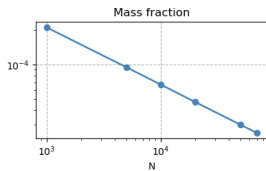
Computations for the Riemann problem

$t = 0$



Convergence

Relative L^1 error at $t = 0.1$



Perspectives

- Try simulations for different pressure laws.
- Incorporate a friction term in the momentum equation.
- Design a two-dimensional numerical scheme.

Thank you for your attention!

References

- [BCT23] Cosmin Burtea, Timothée Crin-Barat, and Jin Tan. “Pressure-relaxation limit for a one-velocity Baer–Nunziato model to a Kapila model”. In: *Mathematical Models and Methods in Applied Sciences* (2023).
- [LSS26] Pierrick Le Vourc’h, Khaled Saleh, and Nicolas Seguin. “Averaged models for compressible two-phase stratified flows on thin domains”. 2026.
- [DP99] Donald A. Drew and Stephen L. Passman. *Theory of Multicomponent Fluids*. 1999.
- [SW84] H. Bruce Stewart and Burton Wendroff. “Two-phase flow: Models and methods”. In: *Journal of Computational Physics* (1984).